# 第7章 图

- 7.1 图的基本概念
- 7.2 通路与回路
- 7.3 无向图的连通性
- 7.4 无向图的连通度
- 7.5 有向图的连通性

# 连通(connected)

- 连通(connected): 无向图G=<V,E>,

  u~v $\Leftrightarrow$ u与v之间有通路

# 连通(connected)

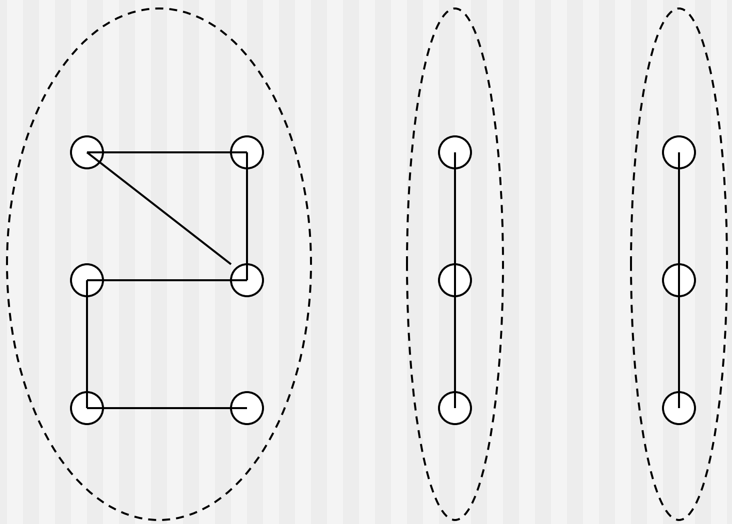
- 规定: $\forall u(u \sim u)$
- 连通关系~是等价关系
  - 自反
  - 对称
  - 传递

# 连通(connected)

- 商集是 $V/\sim = \{ V_1,V_2,\ldots,V_k \}$
- 连通分支(component): $G[V_i]$, $(i=1,\ldots,k)$
- 连通分支数: $p(G)=|V/\sim|=k$

# 连通分支(举例)

- p(G)=3



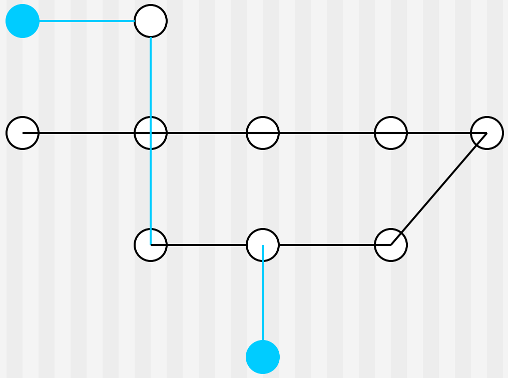
G

# 连通图

- 连通图: $p(G)=1$
- 非连通图: $p(G)>1$

# 短程线(geodesic)

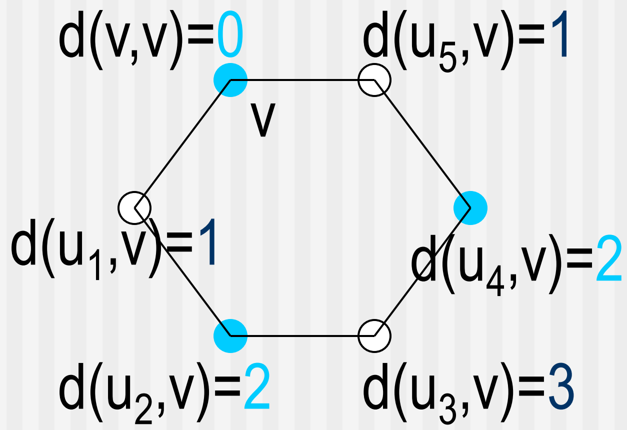
- 短程线：若u,v连通，称u,v之间长度最短的通路



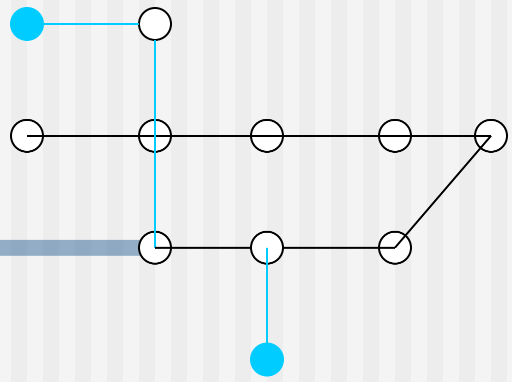
# 距离

- 距离(distance):
  $d_G(u,v)$ = u,v之间短程线的长度
  当u,v不连通时, $d_G(u,v)=\infty$

# 距离举例



# 直径



- 直径(diameter): 顶点之间最大距离

$d(G) = \max\{ d_G(u,v) \mid u,v \in V(G) \}$

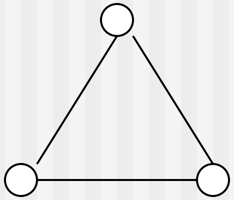
# 短程线,距离,直径

- 短程线: u,v之间长度最短的通路
- 距离(distance):

  $d_G(u,v)$ = u,v之间短程线的长度
- 直径(diameter): 顶点之间最大距离

  $d(G) = \max\{ d_G(u,v) \mid u,v \in V(G) \}$
- 例: $d(K_n)=1 (n\geq 2)$, $d(C_n)=\lfloor n/2 \rfloor$,

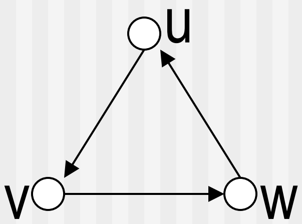
  $d(N_1)=0$, $d(N_n)=\infty\ (n\geq 2)$



# 距离函数

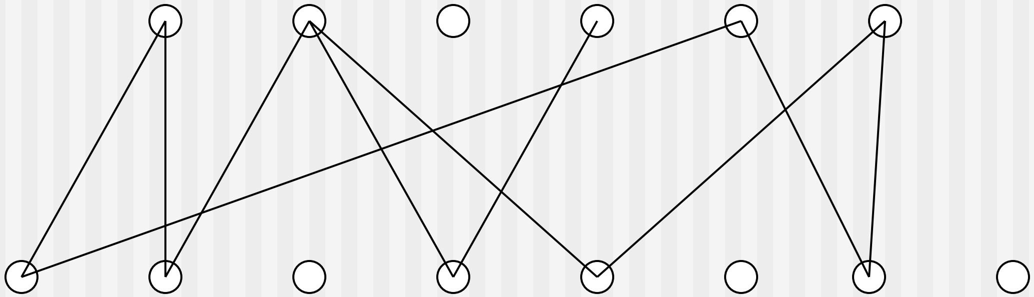
- 非负性: $d(u,v)\geq 0$, $d(u,v)=0 \Leftrightarrow u=v$
- 对称性: $d(u,v)=d(v,u)$
- $\Delta$不等式: $d(u,v)+d(v,w)\geq d(u,w)$
- 任何函数只要满足上述三条性质, 就可以当作距离函数使用
- 无向图的距离函数$d_G(u,v)$满足上述要求
- 有向图的“距离”函数$d_D(u,v)$不对称:

  $d(u,v)=1$, $d(v,u)=2$



# 定理7.8

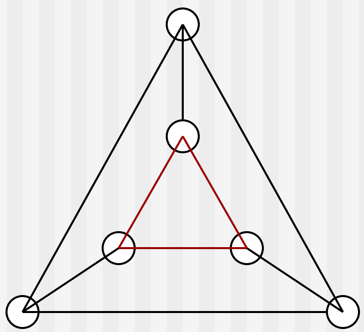
- 定理7.8: G是二部图 ⇔ G中无奇圈



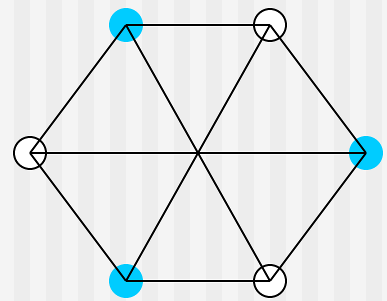
# 定理7.8(证明)

- 定理7.8: G是二部图 $\Leftrightarrow$ G中无奇圈
- 证明: ($\Rightarrow$) 设二部图$G=<V_1,V_2,E>$, 设C是G中任意圈, $C=v_1v_2\ldots v_{k-1}v_kv_1$. 不妨设$v_1\in V_1$, 则$v_3,v_5,\ldots,v_{k-1}\in V_1$, $v_2,v_4,\ldots,v_k\in V_2$, 所以k是偶数. $|C|=k$, C是偶圈.

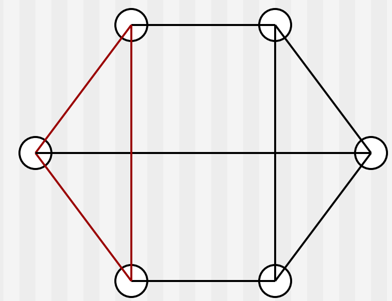
# 示意



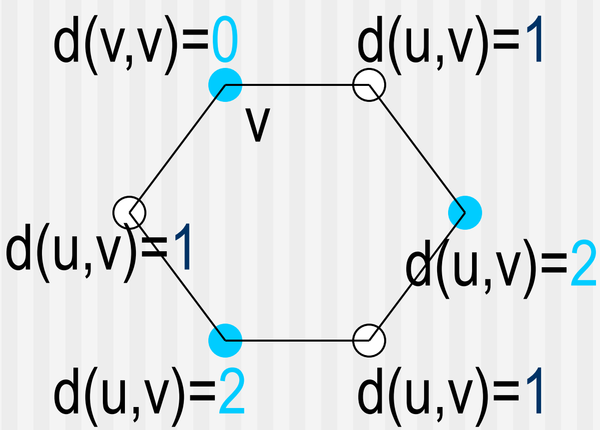
$G_1$

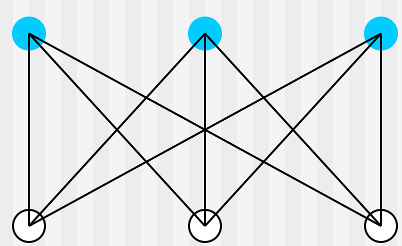


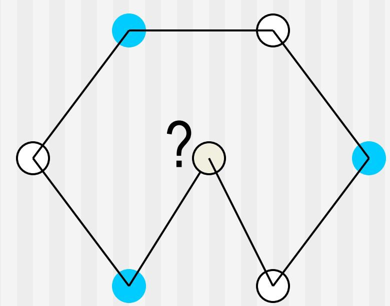
$G_2$



$G_3$






|C|=7

# 定理7.8(续)

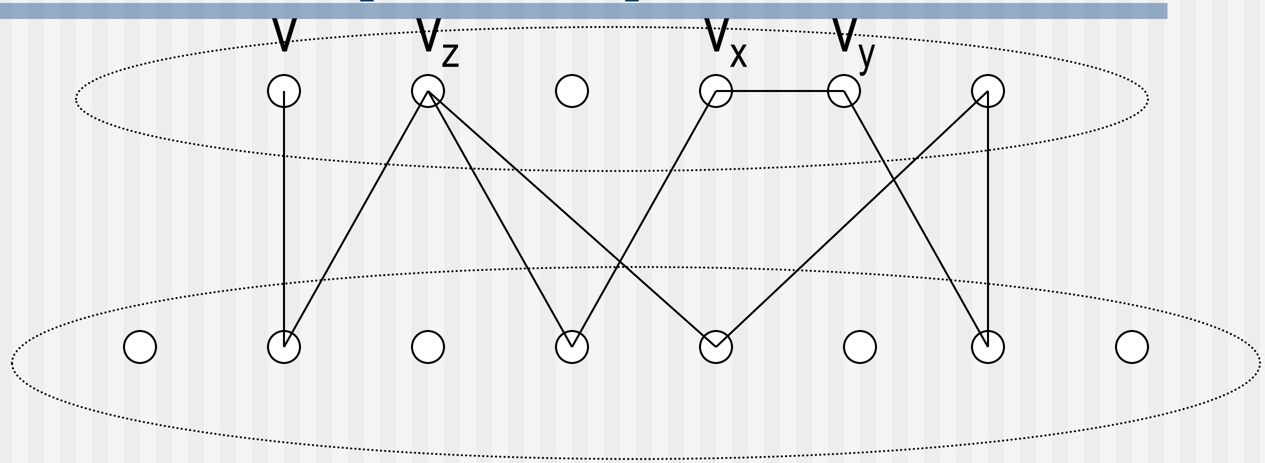
- 定理7.8: G是二部图 $\Leftrightarrow$ G中无奇圈
- 证明: ($\Leftarrow$)设G中无奇圈.设G连通,否则对每个连通分支进行讨论. $\forall v \in V(G)$, 令

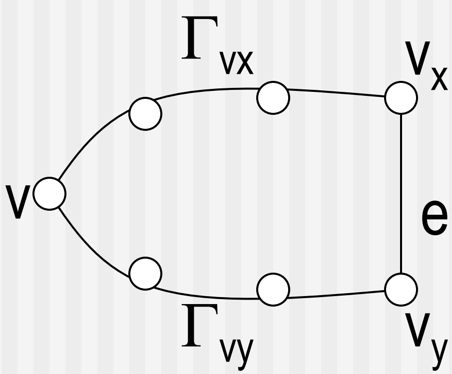
$$V_1 = \{\ u \mid u \in V(G) \wedge d(u,v)\text{是偶数}\ \}$$

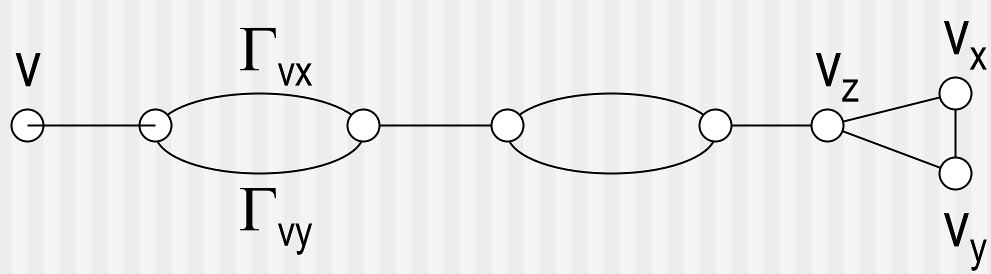
$$V_2 = \{\ u \mid u \in V(G) \wedge d(u,v)\text{是奇数}\ \}$$

则$V_1 \cap V_2 = \varnothing$, $V_1 \cup V_2 = V(G)$. 下面证明 $E \subseteq V_1 \& V_2$,即任意$e \in E(G)$,一个端点在$V_1$,另一个端点在$V_2$.

# 定理7.8(示意)



# 定理7.8(续)

- 证明(续): (⇐) 下面证明 $E \subseteq V_1 \& V_2$.

(反证) 设$e=(v_x,v_y) \in E$, $v_x,v_y$均属于$V_1$

设$\Gamma_{vx}$是v到$v_x$的短程线,$\Gamma_{vy}$是v到$v_y$的短程线, 则 $|\Gamma_{vx}|=d(v,v_x)$与$|\Gamma_{vy}|=d(v,v_y)$均为偶数.

设$v_z$是$\Gamma_{vx}$与$\Gamma_{vy}$的靠近$v_x$与$v_y$一侧的最后一个公共点,则 $|\Gamma_{zx}|$与$|\Gamma_{zy}|$同奇偶

( 因为 $d(v,v_x) = d(v,v_z) + d(v_z,v_x)$,

$d(v,v_y) = d(v,v_z) + d(v_z,v_y)$. 短程线! ).

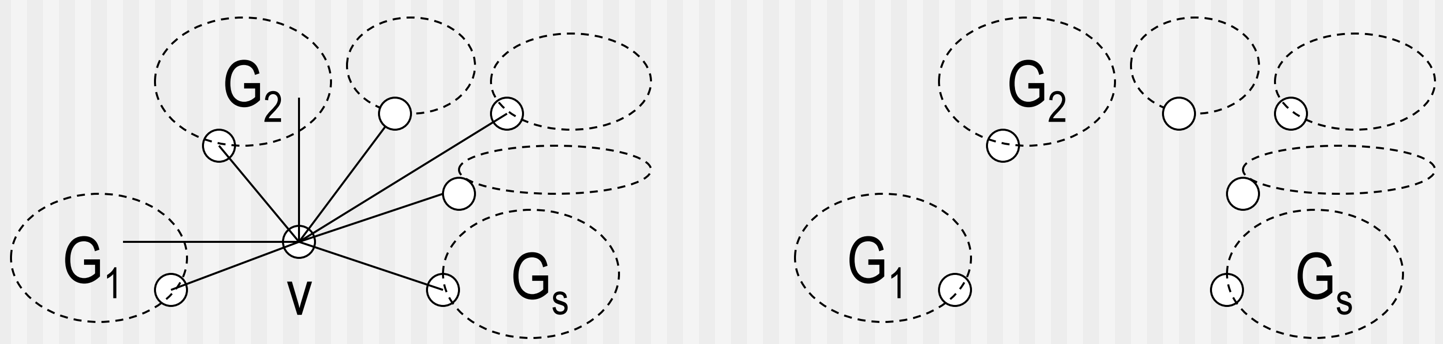
所以$\Gamma_{zx} \cup e \cup \Gamma_{zy}$是奇圈,矛盾! #

# 定理7.9

- 定理7.9: 若n阶无向图G是连通的,则G的边数 $m \geq n-1$.
- 证明: 不妨设G是简单图. (对n归纳.)

(1) $G=N_1$: $n=1$, $m=0$.

(2) 设 $n \leq k$ 时命题成立,下证 $n=k+1$ 时也成立.



# 定理7.9(续)

- 证明(续): (2)设$n \le k$时命题成立,下证$n=k+1$时也成立.

  取$\forall v \in V(G)$,$G' = G-v$,设$p(G')=s$, 连通分支分别为$G_1,G_2,\ldots,G_s$,设$|V(G_i)|=n_i$, $|E(G_i)|=m_i$, $(i=1,2,\ldots,s)$,由归纳假设知$m_i \ge n_i-1$.又由于删除v产生s个连通分支,所以至少删除了s条边,即$d_G(v) \ge s$, 则

$$\begin{aligned} m &= m_1+m_2+\ldots+m_s+d_G(v) \\ &\ge (n_1-1)+(n_2-1)+\ldots+(n_s-1)+s \\ &= n_1+n_2+\ldots+n_s = n-1. \qquad \# \end{aligned}$$

# 可达(reachable)

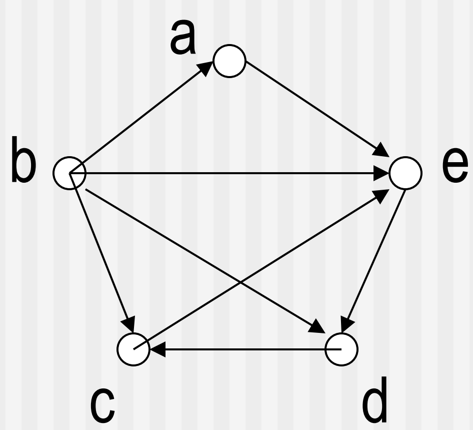
- 可达: 有向图D中,若从顶点u到v存在通路, 则称u可达v,记作$u \to v$
- 规定: $\forall u(u \to u)$,
- 可达关系是自反,传递的

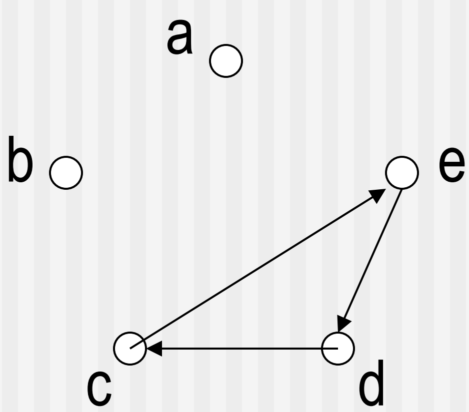
# 可达(reachable)

- 相互可达(双向可达): $u \leftrightarrow v \Leftrightarrow u \rightarrow v \wedge v \rightarrow u$
- 双向可达关系是等价关系

# 可达(举例)

- 可达:
  - b→a→e→d→c
  - c→e→d
- 双向可达：c↔e↔d



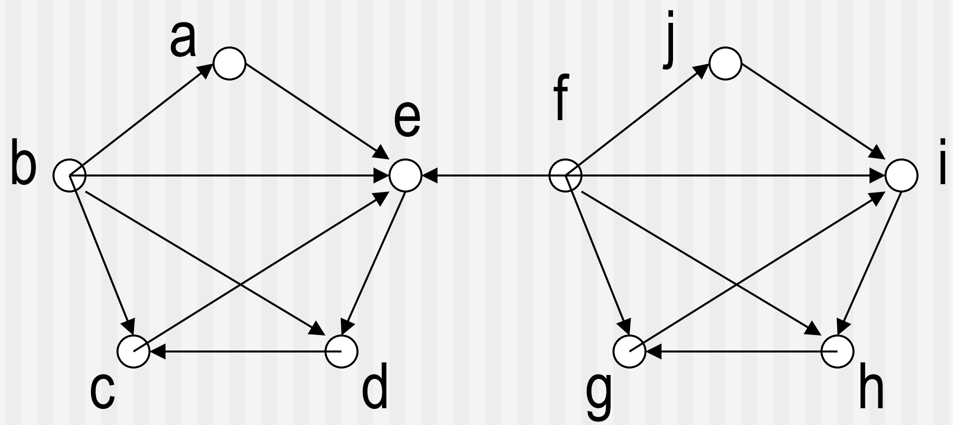



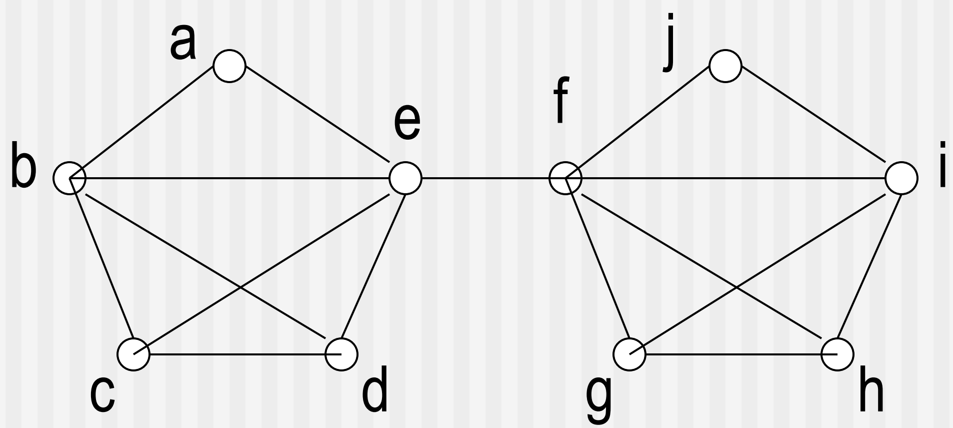
# 短程线,距离

- 短程线：若u→v,u到v长度最短的通路
- 距离：短程线的长度,d<u,v>

# 弱连通(weakly connected)

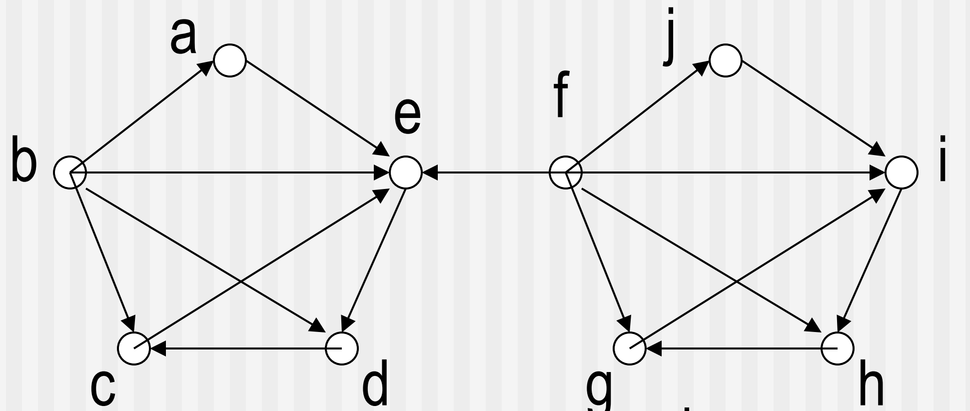
- 弱连通：有向图的基图是连通图



# 单向连通

- 单向连通：有向图的任何一对顶点之间至少单向可达

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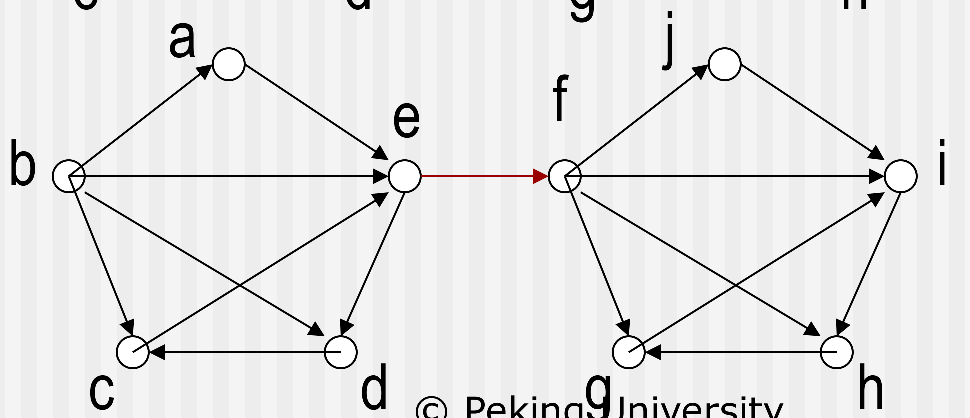
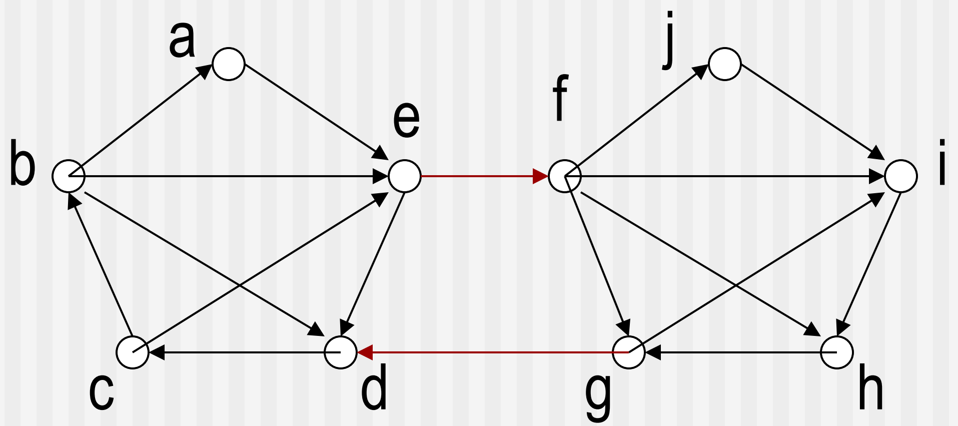



√

# 强连通(strongly connected)

- 强连通(双向连通)：有向图的任何一对顶点之间都相互可达

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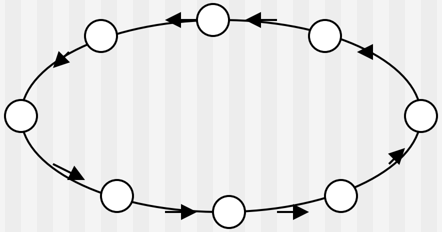
# 定理7.21:强连通的充要条件

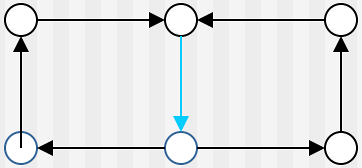
- 定理7.21: 有向图D强连通 ⇔ D中有回路过每个顶点至少一次.



- 证明: (⇐) 显然

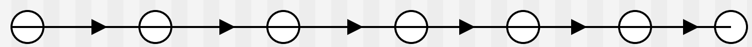
(⇒) 设$V(D)=\{v_1,v_2,\ldots,v_n\}$, 设$\Gamma_{i,j}$是从$v_i$到$v_j$的有向通路, 则$\Gamma_{1,2}+\Gamma_{2,3}+\ldots+\Gamma_{n-1,n}+\Gamma_{n,1}$是过每个顶点至少一次的回路. #

- 说明: 不一定有简单回路,反例如下:

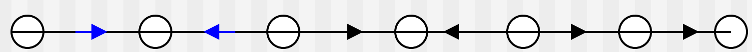


# 定理7.22:单向连通的充要条件

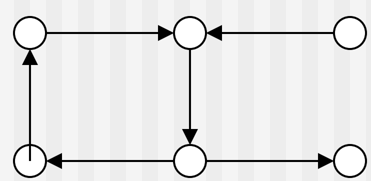
- 定理7.22: 有向图D单向连通 ⇔ D中有通路过每个顶点至少一次.
- 证明: (⇐) 显然



  (⇒) ?

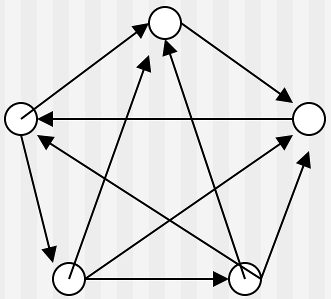


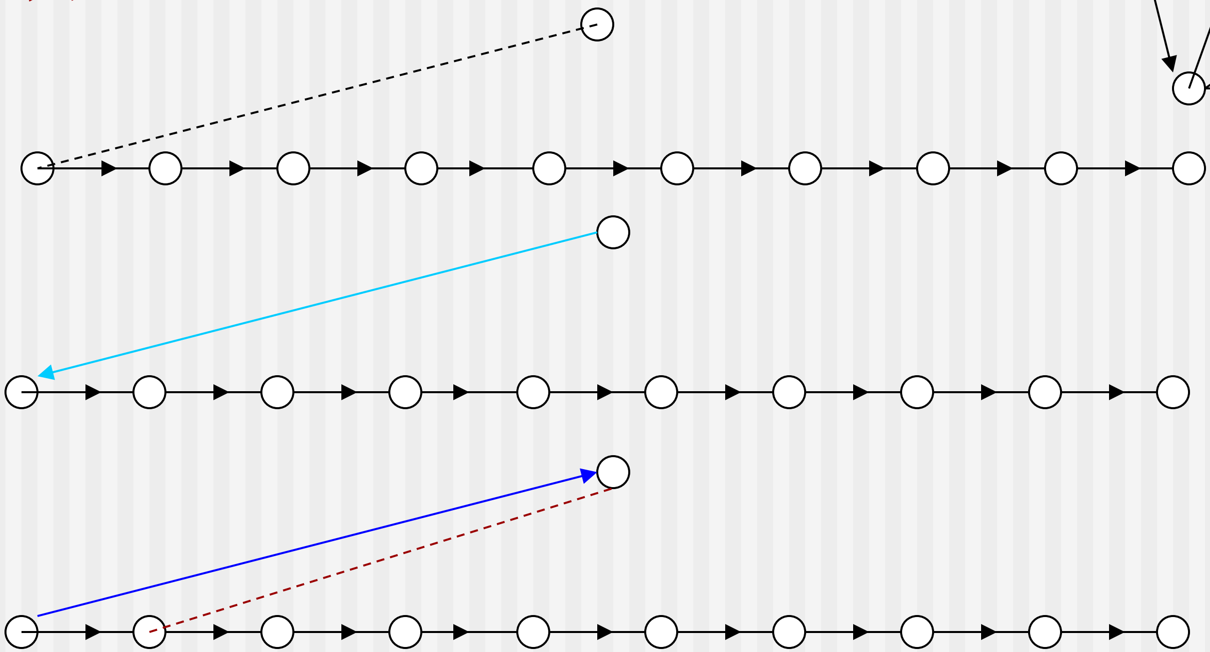
- 说明: 不一定有简单通路,反例如下:



# 命题

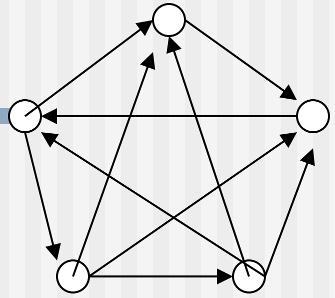
- 命题：竞赛图一定有初级通路(路径)过每个顶点恰好一次
- 证明:

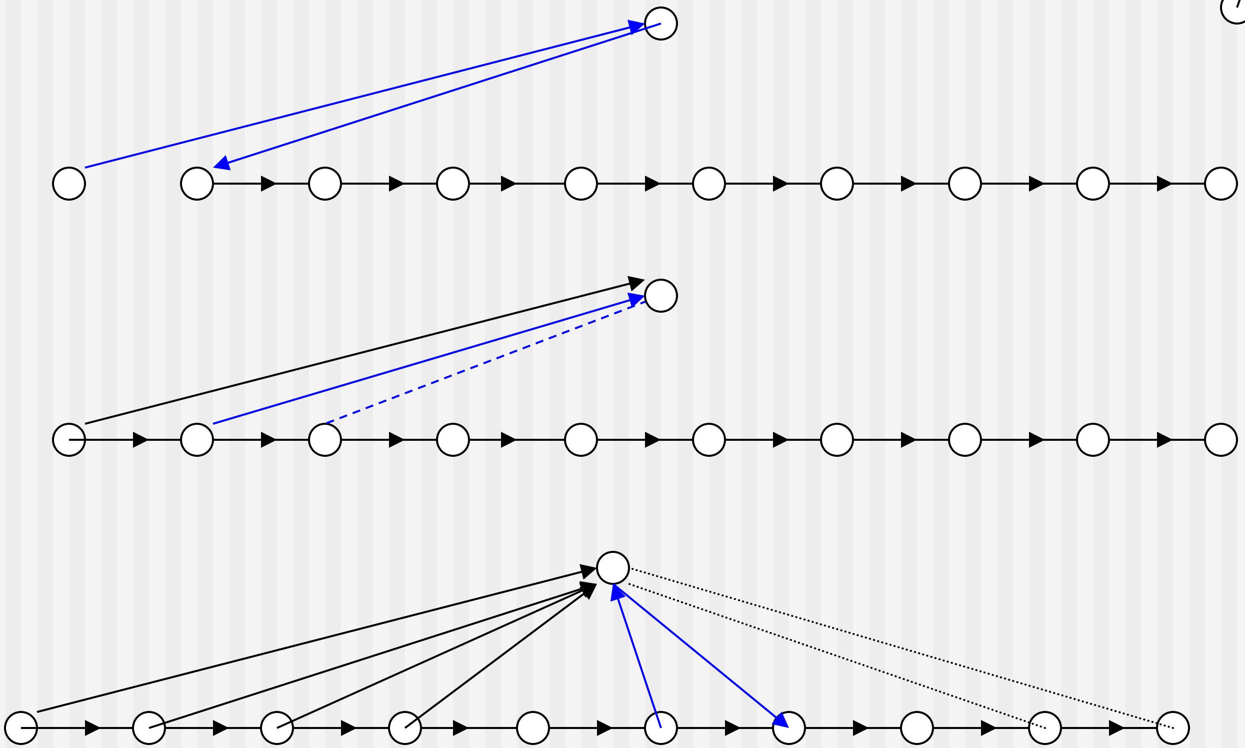




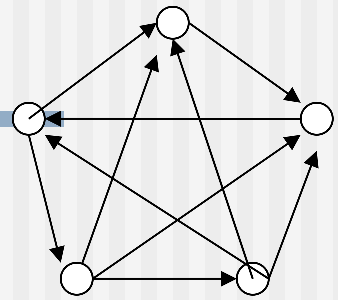
# 命题(续)

- 证明(续):

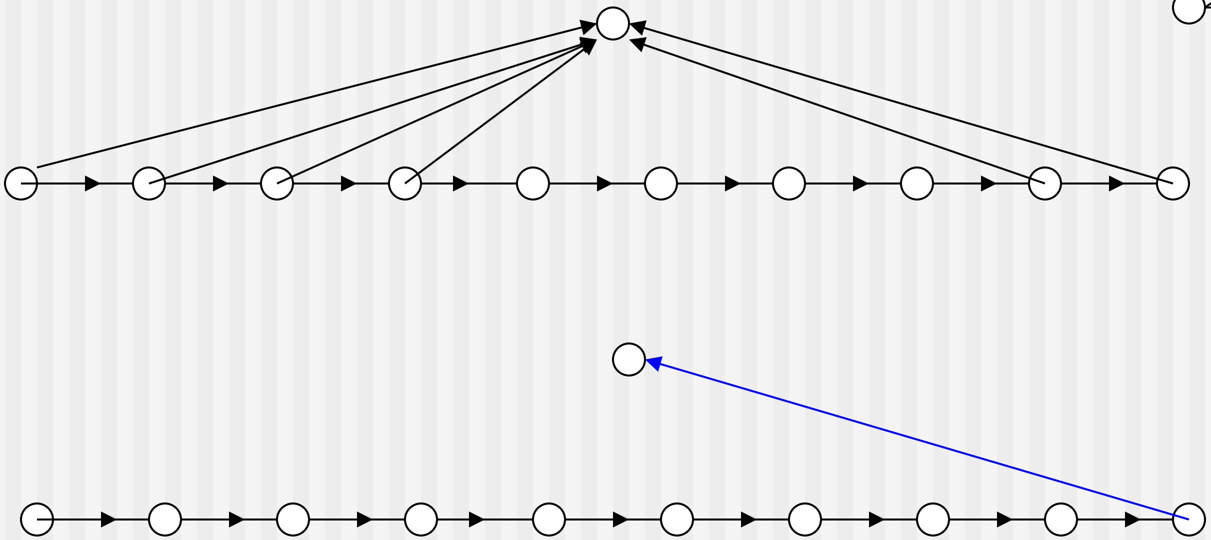




# 命题(续)



- 证明(续):



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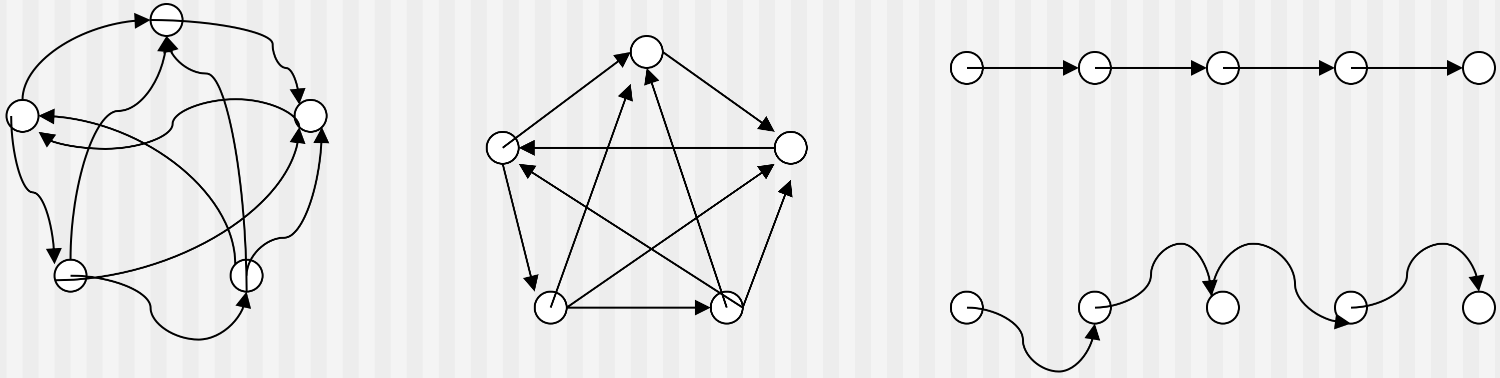
# 命题(续)

- 意义：竞赛图可给n个选手完全排定名次，上述通路的两个端点是“第一”与“最末”
- 用途：证明定理7.22



# 定理7.22(续)

- 定理7.22: 有向图D单向连通 $\Leftrightarrow$ D中有通路过每个顶点至少一次.
- 证明: ($\Rightarrow$)设$V(D)=\{v_1,v_2,\ldots,v_n\}$,则在任意一对顶点$v_i$与$v_j$之间至少有一条单向通路.

  定义V上的竞赛图D',使得顶点$v_i$与$v_j$之间D'边的方向与D中单向通路方向一致.
  根据命题,得到D'中路径过每个顶点.

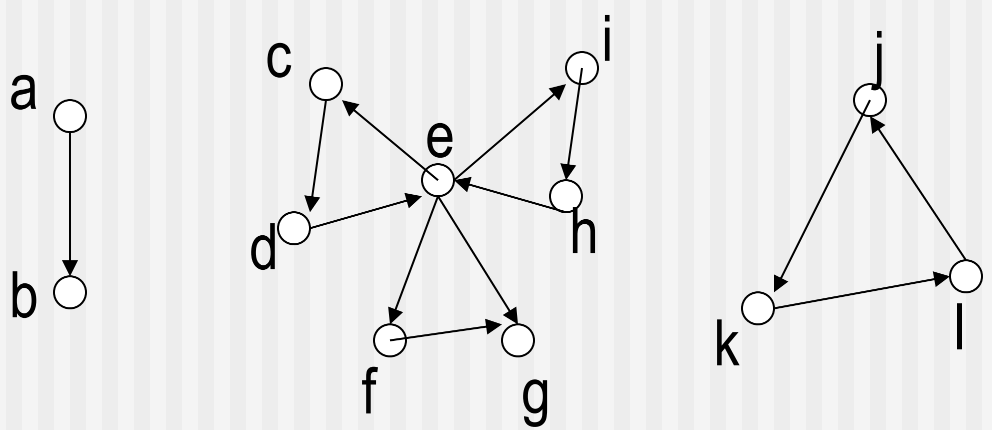
  把D中单向通路逐段"代入"上述D'中路径, 即得到所求通路. #

# 有向图的连通分支

- 强连通分支(strong component): 极大强连通子图
- 单向连通分支: 极大单向连通子图
- 弱连通分支(weak component): 极大弱连通子图

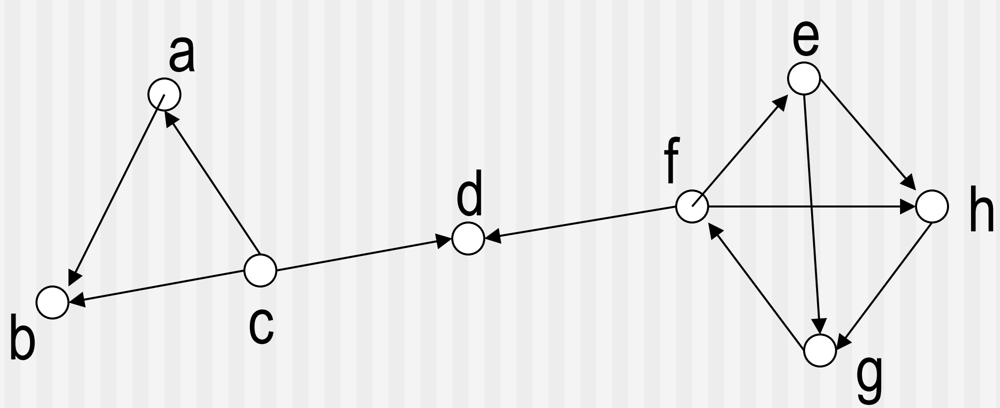
# 连通分支(例7.8)

- 强连通分支:
  G[{a}],G[{b}],G[{c,d,e,h,i}],
  G[{f}],G[{g}],G[{j,k,l}]
- 单向连通分支:
  G[{a,b}],G[{c,d,e,h,i,f,g}], G[{j,k,l}]
- (弱)连通分支: 与单向连通分支相同



# 连通分支(例7.8(b))

- 强连通分支: G[{a}], G[{b}], G[{c}], G[{d}], G[{e,f,g,h}]
- 单向连通分支: G[{a,b,c}], G[{c,d}], G[{d,e,f,g,h}]
- (弱)连通分支: G



# 总结

- 连通图，连通分支，连通分支数
- 二部图的判别定理
- 弱连通,单向连通,强连通(双向连通)

# 作业

- P131: 14， 16